

Combinations

Lecture 46

Section 9.5

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- 1 Combinations
- 2 Poker Hands
- 3 Permutations with Repeated Elements
- 4 Partitions into r Subsets
- 5 Assignment

Outline

- 1 Combinations
- 2 Poker Hands
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Combinations

Definition

Let n and r be nonnegative integers with $r \leq n$. An r -combination of a set of n elements is a subset of r of the n elements.

Theorem

The number of r -combinations of a set of n elements is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Combinations

Proof.

- Let C be the number of r -combinations of r elements of a set of n elements.
- Each subset of r elements may be permuted in $r!$ ways, giving $C \cdot r!$ permutations.
- But we already know that the number of r -permutations of r elements from a set of n elements is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

- Therefore, C is

$$\frac{n!}{r!(n-r)!}.$$



Example

- Problems:

- (a) If we have 10 basketball players, in how many ways can they be divided into 2 teams?
- (b) If we have 12 basketball players, in how many ways can they be divided into 2 teams with 2 left over?
- (c) If we have 20 basketball players, in how many ways can they be divided into 4 teams?
- (d) If we have 12 basketball players and 2 of them insist on being on the same team, or on no team, in how many ways can we form two teams?
- (e) If we have 12 basketball players and 2 of them insist on being on different teams, or on no team, in how many ways can we form two teams?

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Poker Hands

- The total number of possible poker hands is $\binom{52}{5} = 2598960$.
- What is the probability of a full house (2 of one kind, 3 of another kind)?
- Consider the sequence:
 - Choose the first kind: 13 ways.
 - Choose the second kind: 12 ways.
 - Choose 2 cards of first kind: $\binom{4}{2} = 6$.
 - Choose 3 cards of second kind: $\binom{4}{3} = 4$.
- Total number of ways: $13 \cdot 12 \cdot 6 \cdot 4 = 3744$.
- Probability of a full house is $\frac{3744}{2598960} = 0.00144$.

Poker Hands

- An alternate way to work the problem.
- Assume the cards are arranged with the two of a kind first, then the three of a kind.
 - The first card can be any card: 52 ways.
 - The second card must be the same kind: 3 ways.
 - The third card must be a different kind: 48 ways.
 - The fourth card must be the same kind: 3 ways.
 - The fifth card must be the same kind: 2 ways.
- That gives $52 \cdot 3 \cdot 48 \cdot 3 \cdot 2 = 44928$ ways.
- Why the different answer?

Poker Hands

- We counted each possible pair twice and we counted each possible triple $3! = 6$ times.
- To correct for that, we must divide.
- The answer is $\frac{44928}{2 \cdot 6} = 3744$.

Poker Hands

- Find the probabilities of
 - No pairs.
 - One pair.
 - Three of a kind.
 - Four of a kind.
 - Two pairs.
- Which is most likely?
- Which is least likely?

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Permutations with Repeated Elements

- There is a lake in Massachusetts named Lake Chaubunagungamaug.
- (The longer form of the name is Lake Chargoggagoggmanchauggagoggchaubunagungamaugg.)
- In how many ways can we arrange the letters in the word Chaubunagungamaug?
- The word contains
 - 1 C, 1 H, 1 B, and 1 M.
 - 2 N's.
 - 3 G's.
 - 4 A's and 4 U's.

Permutations with Repeated Elements

- One method.
 - There are $17! = 355687428096000$ permutations of 17 letters.
 - However, the 4 A's could be permuted in $4! = 24$ ways without making a difference.
 - The same is true for the other letters.
 - The number of *distinct* permutations is $\frac{17!}{4!4!3!2!} = 51459408000$.
- Another method.
 - There are $\binom{17}{4} = 715$ places to put the A's.
 - Then there are $\binom{13}{4} = 126$ places to put the U's.
 - And so on.
 - The total number is $\binom{17}{4} \binom{13}{4} \binom{9}{3} \binom{6}{2} \cdot 4! = 51459408000$.

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Partitions into r Subsets

Definition

Let n and r be positive integers with $1 \leq r \leq n$. The **Stirling number** (of the second kind) $S_{n,r}$ is the number of ways in which a set of n elements can be partitioned into r subsets.

- Clearly, $S_{n,1} = 1$ and $S_{n,n} = 1$.

Example

- Find $S_{5,2}$, $S_{5,3}$, and $S_{5,4}$.
- For $S_{5,4}$
 - We must have 3 sets of 1 and 1 set of 2.
 - There are $\binom{5}{2} = 10$ ways to do that, so $S_{5,4} = 10$.

For $S_{5,3}$

- We could have 1 set of 3 and 2 sets of 1, or 2 sets of 2 and one set of 1.
 - There are $\binom{5}{3} = 10$ ways to have the first.
 - There are $\binom{5}{2} \binom{3}{2} \div 2 = 15$ ways to have the second.
 - So $S_{5,3} = 25$.
- Find $S_{5,2}$.

A Recurrence Relation

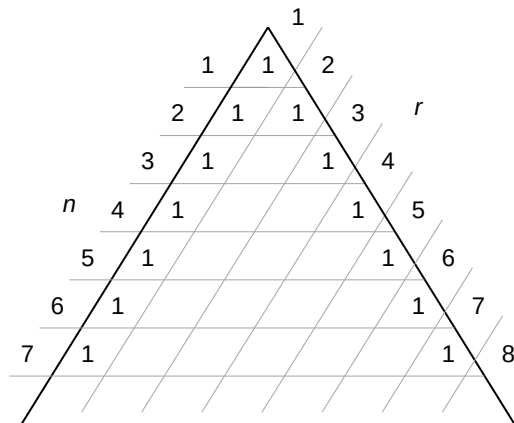
- We will relate $S_{n,r}$ to $S_{n-1,r-1}$ through a recurrence relation.
- Remove an element x to leave a set of $n - 1$ elements.
- Consider the $S_{n-1,r-1}$ partitions of the $n - 1$ elements into $r - 1$ subsets.
- If we reintroduce x as a singleton set $\{x\}$, we get a partition of n elements into r subsets.

Partitions into r Subsets

- Now consider the $S_{n-1,r}$ partitions of the $n - 1$ elements into r subsets.
- Reintroduce the x by adding it to one of the subsets.
- This can be done in r different ways and each way will produce a partition of the n elements into r subsets.
- Thus, the total number of ways of partitioning n elements into r subsets is

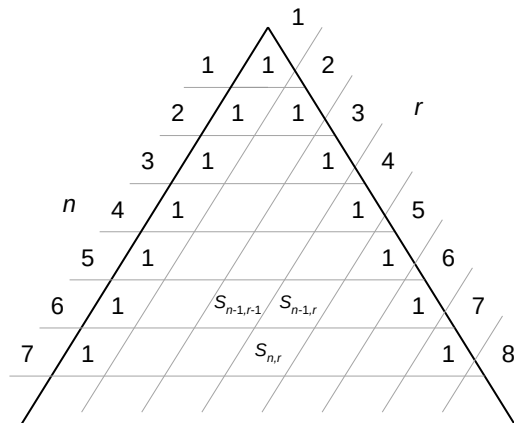
$$S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}.$$

Partitions into r Subsets



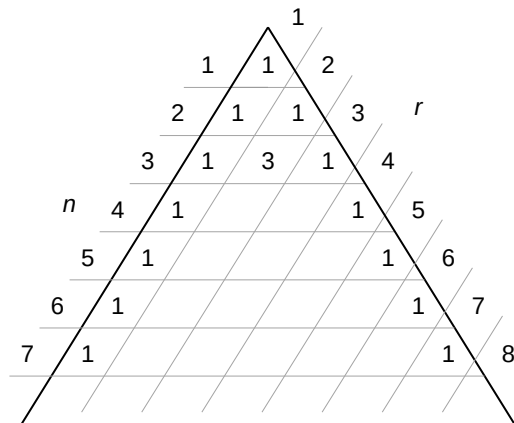
Fill in the boundaries with 1's

Partitions into r Subsets



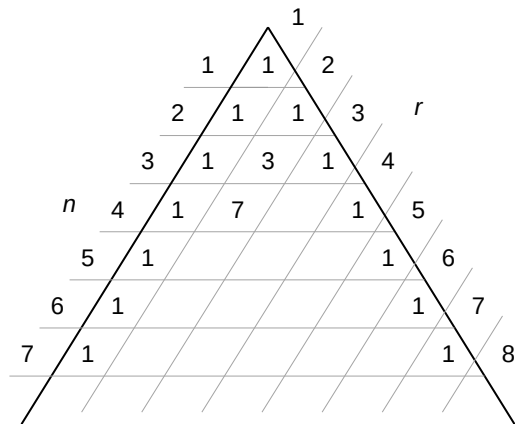
The recursive relation

Partitions into r Subsets



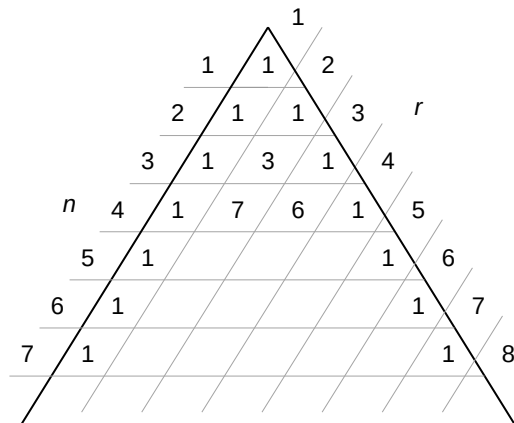
Compute $S_{3,2}$

Partitions into r Subsets



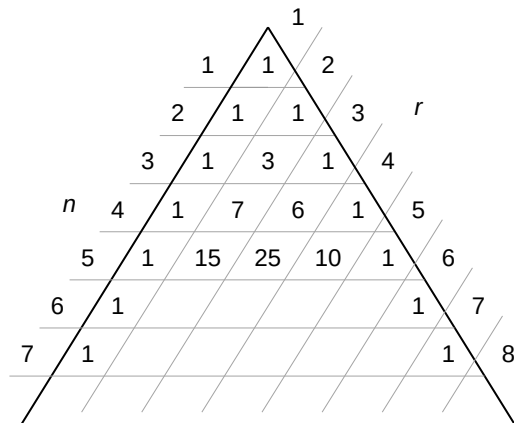
Compute $S_{4,2}$

Partitions into r Subsets



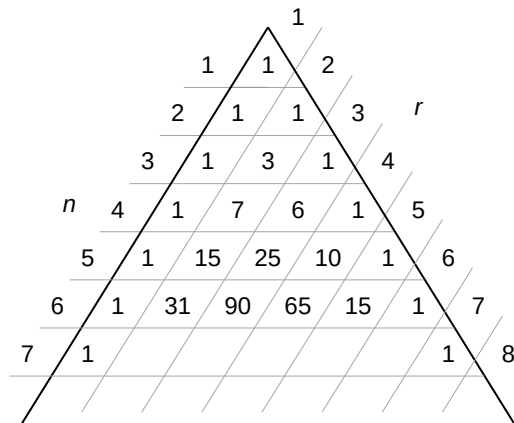
Compute $S_{4,3}$

Partitions into r Subsets



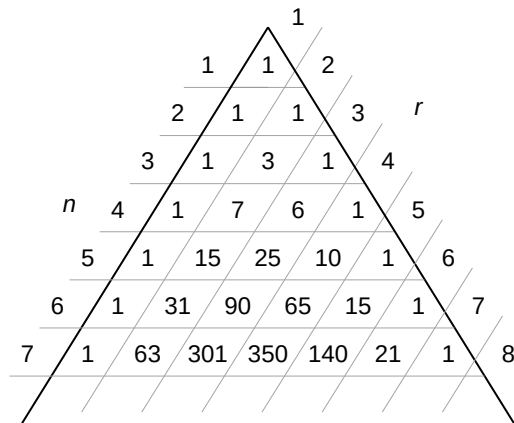
Compute $S_{5,2}$, $S_{5,3}$, and $S_{5,4}$

Partitions into r Subsets



Compute $S_{6,2}$, $S_{6,3}$, $S_{6,4}$, and $S_{6,5}$

Partitions into r Subsets



Compute $S_{7,2}$, $S_{7,3}$, $S_{7,4}$, $S_{7,5}$, and $S_{7,6}$

Example

- A group of 7 campers decide to break up into 3 groups to go on 3 different hikes.
- In how many ways can they divide themselves up?

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Assignment

Assignment

- Read Sections 9.5, pages 565 - 581.
- Exercises 2, 5, 6, 7, 10, 17, 20, 21, 32($S_{5,4}$), 35, page 581.