

# The Binomial Theorem

Lecture 47  
Section 9.7

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- 1 Combinations
- 2 Pascal's Triangle
- 3 The Binomial Theorem
- 4 Binomial Probabilities
- 5 Assignment

# Outline

- 1 Combinations
- 2 Pascal's Triangle
- 3 The Binomial Theorem
- 4 Binomial Probabilities
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# Combinations

## Theorem

Let  $n$  and  $r$  be nonnegative integers with  $r \leq n$ . Then

$$\binom{n}{r} = \binom{n}{n-r}.$$

## Proof.

- To choose which  $r$  elements to include in the subset is the same as choosing which  $n - r$  elements *not* to include.
- Thus,  $\binom{n}{r} = \binom{n}{n-r}$ .



# A Recurrence Relation

## Theorem

Let  $n$  and  $r$  be positive integers with  $r < n$ . Then

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

## Proof.

- Let  $A$  be a set of  $n$  elements and let  $x \in A$ .
- Divide the subsets of size  $r$  into two groups:
  - (1) Those that contain  $x$ .
  - (2) Those that do not contain  $x$ .



# A Recurrence Relation

Proof.

- How many subsets are in group (1)?



# A Recurrence Relation

## Proof.

- How many subsets are in group (1)?
  - If we remove  $x$  from each, we have all possible subsets of  $r - 1$  elements from  $A - \{x\}$ , a set of  $n - 1$  elements.
  - So, there are  $\binom{n-1}{r-1}$  such subsets.



# A Recurrence Relation

## Proof.

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  - So, there are  $\binom{n-1}{r-1}$  such subsets.
- How many subsets are in group (2)?



# A Recurrence Relation

## Proof.

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  - If we remove  $x$  from each, we have all possible subsets of  $r - 1$  elements from  $A - \{x\}$ , a set of  $n - 1$  elements.
  - So, there are  $\binom{n-1}{r-1}$  such subsets.
- How many subsets are in group (2)?
  - The element  $x$  is in none of them, so if we remove  $x$  from  $A$ , these subsets are all possible subsets of  $r$  elements from  $A - \{x\}$ , a set of  $n - 1$  elements.
  - So, there are  $\binom{n-1}{r}$  such subsets.



# A Recurrence Relation

## Proof.

- How many subsets are in group (1)?
  - If we remove  $x$  from each, we have all possible subsets of  $r - 1$  elements from  $A - \{x\}$ , a set of  $n - 1$  elements.
  - So, there are  $\binom{n-1}{r-1}$  such subsets.
- How many subsets are in group (2)?
  - The element  $x$  is in none of them, so if we remove  $x$  from  $A$ , these subsets are all possible subsets of  $r$  elements from  $A - \{x\}$ , a set of  $n - 1$  elements.
  - So, there are  $\binom{n-1}{r}$  such subsets.
- Thus,  $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ .



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# Pascal's Triangle

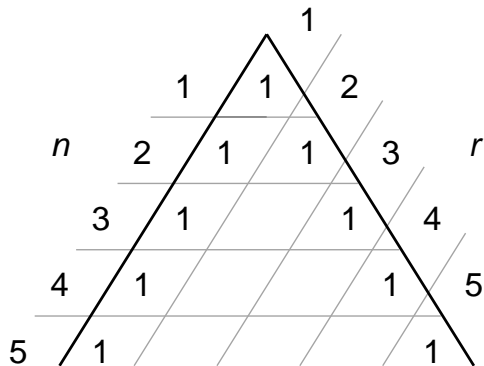
- The equation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

allows us to compute  $\binom{n}{r}$  recursively.

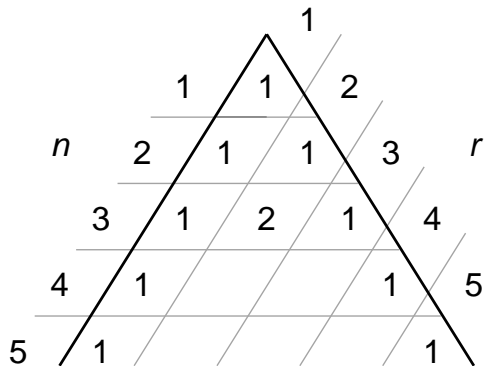
- The recursion ends with the boundary cases  $\binom{n}{0} = 1$  and  $\binom{n}{n} = 1$ .
- This is the basis of **Pascal's Triangle**.

# Pascal's Triangle



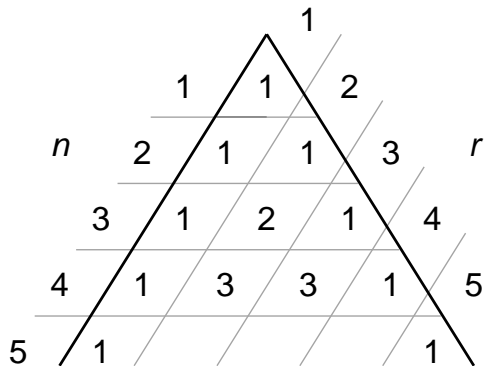
Initialize the boundary to 1

# Pascal's Triangle



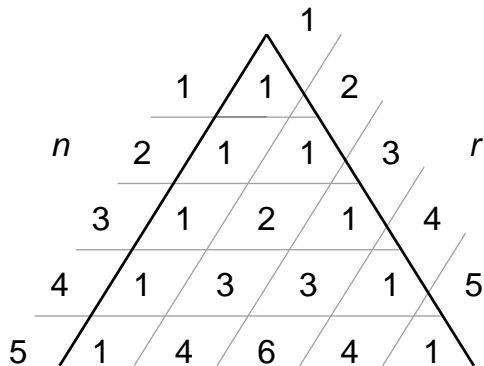
Compute  $\binom{3}{2}$

# Pascal's Triangle



Compute  $\binom{4}{2}$  and  $\binom{4}{3}$

# Pascal's Triangle



Compute  $\binom{5}{2}$ ,  $\binom{5}{3}$ , and  $\binom{5}{4}$

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# The Binomial Theorem

## Theorem

Let  $n$  be a nonnegative integer and let  $a$  and  $b$  be any real numbers. Then

$$\begin{aligned}(a + b)^n &= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a b^{n-1} + b^n \\ &= \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i.\end{aligned}$$

# The Binomial Theorem

## Proof.

- The proof is by induction on  $n$ .
- When  $n = 0$ , we have  $(a + b)^0 = 1$  and

$$\begin{aligned}\sum_{i=0}^0 \binom{n}{i} a^{n-i} b^i &= \binom{0}{0} a^{0-0} b^0 \\ &= 1.\end{aligned}$$

- Therefore, the statement is true when  $n = 0$ .



# The Binomial Theorem

## Proof.

- Suppose that the statement is true for some integer  $k$  where  $k \geq 0$ .
- Then

$$\begin{aligned}(a + b)^n &= (a + b)(a + b)^{n-1} \\ &= (a + b) \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-1-i} b^i \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-i} b^i + \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-i} b^{i+1} \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} a^{n-i} b^i + \sum_{i=1}^n \binom{n-1}{i-1} a^{n-1-i} b^i\end{aligned}$$



# The Binomial Theorem

## Proof.



$$\begin{aligned} &= a^n + \sum_{i=1}^{n-1} \binom{n-1}{i} a^{n-i} b^i + \sum_{i=1}^{n-1} \binom{n-1}{i-1} a^{n-1-i} b^i + b^n \\ &= a^n + \sum_{i=1}^{n-1} \left[ \binom{n-1}{i} + \binom{n-1}{i-1} \right] a^{n-i} b^i + b^n \\ &= a^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} b^i + b^n \\ &= \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i. \end{aligned}$$

- Therefore, the statement is true when  $n = k + 1$ , and so it is true for all  $n \geq 0$ .



# Examples

- Expand  $(a + b)^5$ .
- Expand  $(a - b)^5$ .
- Expand  $(a + 2b)^5$ .
- Show that  $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ .
- Show that  $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n} = 0$ .
- What is the value of  $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n}$ ?
- What is the value of  $\binom{n}{0} - 2\binom{n}{1} + 2^2\binom{n}{2} - \cdots \pm 2^n\binom{n}{n}$ ?

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# Bernoulli Trials

## Definition (Bernoulli Trial)

A **Bernoulli trial** is an experiment that has exactly two possible outcomes. The outcomes are called **success** and **failure**.

- Toss a coin. Outcomes: heads or tails.
- Roll a die. Outcomes: even or odd.
- Draw a card. Outcomes: ace or not ace.

# Binomial Experiments

## Definition (Binomial Experiment)

A **binomial experiment** is an experiment that consists of a fixed number of independent and identical Bernoulli trials. Let  $n$  be the number of trials and let  $p$  be the probability of success.

## Definition (Binomial Random Variable)

A **binomial random variable** is a variable whose value is the number of successes in a binomial experiment.

# Examples

- Toss a coin 10 times. Let  $X$  be the number of heads.
- Roll a die 6 times. Let  $X$  be the number of even rolls.
- Draw 4 cards. Let  $X$  be the number of aces. (Is this binomial?)

# Binomial Probability Distribution

## Proof.

- There are exactly  $\binom{n}{r}$  distinct patterns of successes and failures in the  $n$  trials.
- Each pattern has the same probability, namely,  $p^r(1 - p)^{n-r}$ .
- Therefore, the probability of one of the patterns occurring is

$$\binom{n}{r} p^r (1 - p)^{n-r}.$$



# Binomial Probability Distribution

## Theorem

*Let  $X$  be a binomial random variable with parameters  $n$  and  $p$ . Then the probability of exactly  $r$  successes is*

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}.$$

# Examples

- Toss a coin 10 times. What is the probability of exactly 6 heads?
- Roll a die 6 times. What is the probability of getting a 1 or a 2 exactly 3 times?
- Guess at all 25 answers on a multiple-choice test with 5 choices for each answer. What is the probability of scoring between 3 and 7 correct?

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## Collected

- Sec. 9.2: 7, 39bd, 42.
- Sec. 9.3: 12, 22.
- Sec. 9.4: 27.
- Sec. 9.5: 10, 21, 32.

# Assignment

## Assignment

- Read Sections 9.7, pages 592 - 602.
- Exercises 10, 11, 12, 16, 18, 22, 26, 30, 32, 39, 42, page 603.